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APPROXIMATE SOLUTIONS TO OPTIMUM FLIGHT TRAJECTORIES FOR A TURBOJET-POWERED AIRCRAFT

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APPROXIMATE SOLUTIONS TO OPTIMUM FLIGHT TRAJECTORIES FOR A TURBOJET-POWERED AIRCRAFT

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SUMMARY

The climbing program of a turbojet-powered aircraft is analyzed with respect to minimum time, minimum fuel consumption, and minimum horizontal distance trajectories. By means of the indirect methods of the calculus of variations it is shown that, in the case where the centripetal acceleration effects are neglected, the totality of extremal arcs consists of a number of constant path inclination subarcs and a number of variable path inclination subarcs.

Under suitable hypotheses for the drag function and thrust function, solutions in a closed form are obtained for the variable path inclination subarc. These solutions represent a considerable improvement with respect to the results supplied by the so-called energy-height method, commonly used by aircraft manufacturers. The latter is a graphical-analytical procedure, according to which the speed for best climb is determined as the velocity maximizing the power excess (available power minus required power) for constant value of the energy height $\left(h_e = h + \frac{V^2}{2g}\right)$. With the energy-height method several nonoptimum points must be analyzed, for each value of h_e, as a preliminary step toward the finding of the optimum operating condition. With the present solutions, on the contrary, the optimum operating point is supplied by straightforward computational procedure.

For particular types of drag polars, the variable path inclination subarc may consist of several branches, one of which is subsonic, one transonic, and one supersonic. With regard to minimum time and minimum fuel consumption trajectories, only the subsonic and the supersonic branches are of interest.

Numerical examples are presented for the minimum time problem, and the effect of wing loading on the solutions is investigated.

INTRODUCTION

In reference 1, which presents extensive bibliographical information, trajectories of minimum time were investigated with approximate methods for a rocket-powered aircraft.

In the present analysis a more general category of problems is considered, namely problems involving either time or fuel consumed or horizontal distance flown by the aircraft. Paths extremizing any one of the above three quantities, for the case where each of the other two W
is either free of choice or prescribed, are investigated. 1
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A turbojet-powered aircraft is considered in connection with 7
flight trajectories of relatively short duration. As a consequence, the weight of the airplane is regarded as a constant in the equations of motion.

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SYMBOLS

a	speed of sound, ft sec^{-1}
c	specific fuel consumption, sec^{-1}
C_D	drag coefficient
C_L	lift coefficient
D	drag, lb
D_i	induced drag, lb
E	excess function defined by equation (8), lb
f	function defined by equation (19)
F	fundamental function defined by equation (20)

g	acceleration of gravity, ft sec^{-2}
h	flight altitude, ft
h_e	energy height, $h + \frac{V^2}{2g}$, ft
I	functional form defined by equation (17)
K	ratio of induced drag coefficient to square of lift coefficient
$K_1 \ldots K_3$	numerical constants
L	lift, lb
m	quantity defined by equation (40)
M	Mach number
p	atmospheric pressure, lb ft^{-2}
q	weight of fuel consumed per unit time, lb sec^{-1}
Q	weight of fuel consumed, lb
R	air constant, ft^2 sec^{-2} $^{\circ}$R^{-1}
S	reference surface, ft^2
t	time, sec
T	thrust, lb
V	flight velocity, ft sec^{-1}
W	weight of aircraft, lb
x	exponent appearing in thrust relationship defined by equation (45)
X	horizontal distance flown, ft
y	exponent appearing in specific fuel consumption relationship defined by equation (46)
α	derivative of air temperature with respect to altitude, $^{\circ}$R ft^{-1}

γ ratio of specific heat at constant pressure to specific heat at constant volume

θ path inclination with respect to horizontal plane (positive for climbing)

λ constant defined by equation (44)

μ variable Lagrange multiplier

π ratio of pressure at altitude h to pressure at the tropopause h_*

σ relative density of air

ρ absolute density of air, lb ft^{-4} sec^2

τ absolute temperature of air, $^{\circ}$R

Φ function defined by equation (6), ft^{-2} sec^3

Ψ function defined by equation (7), ft^{-2} sec^2

ω function defined by first member of equation (58)

Superscript:

$(\;)'$ derivative with respect to velocity

Subscripts:

i initial point

f final point

o zero-lift condition, condition at $M = 0$, or sea-level condition

oo zero-lift condition evaluated at $M = 0$

* condition at the tropopause

FUNDAMENTAL HYPOTHESES AND EQUATIONS OF MOTION

The following hypotheses are used throughout the paper:

(1) The turbojet-powered aircraft is regarded as a particle.

(2) The small angle between the thrust vector $\overline{T}$ and the velocity vector $\overline{V}$ is neglected.

(3) Aerodynamic lag is disregarded, that is, lift L and drag D forces are calculated as in unaccelerated flight.

(4) Weight W is regarded as a constant.

(5) The centrifugal component of acceleration is not considered and the equation of motion on the normal to the flight path is approximated (ref. 1) as

$$L - W = 0 \tag{1}$$

(6) Only flight paths contained in a vertical plane are considered.

(7) The propulsion system is operated in such a way that thrust T, specific fuel consumption c, and fuel consumed per unit time q are known functions of velocity V and altitude h

$$T = T(V,h) \tag{2}$$

$$c = c(V,h) \tag{3}$$

$$q = Tc = q(V,h) \tag{4}$$

In the light of the above hypotheses, the equation of motion on the tangent to the flight path is written (after simple transformations) as

$$\frac{h'}{V \sin \theta} - \Phi - \psi h' = 0 \tag{5}$$

$$\Phi = \frac{W}{Eg} \tag{6}$$

$$\psi = \frac{W}{EV} \tag{7}$$

$$E = T - D \tag{8}$$

$$h' = \frac{dh}{dV} \tag{9}$$

the E function is denominated the excess function.

With regard to the drag function, hypothesis (3) leads to a general expression of the form

$$D = D_o(h,V) + D_i(h,V,L) \tag{10}$$

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where D_o is the zero lift drag. After accounting for equation (1) and hypothesis (4), one concludes that the D function can be written in the form

$$D = D(V,h) \tag{11}$$

Because of equation (2) the E function has also the form $E = E(V,h)$. As a consequence, both the Φ function and the ψ function can be regarded as obeying relationships of the type

$$\Phi = \Phi(V,h) \tag{12}$$

$$\psi = \psi(V,h) \tag{13}$$

FORMULATION OF LAGRANGE PROBLEM

The time t necessary to transfer the turbojet-powered aircraft from an initial condition of flight i to a final condition of flight f is given by

$$t = \int_{V_i}^{V_f} \frac{h'}{V \sin \theta} dV = \int_{V_i}^{V_f} (\Phi + \psi h')dV \tag{14}$$

The fuel consumed Q is supplied by

$$Q = \int_{V_i}^{V_f} q \frac{h'}{V \sin \theta} \, dV = \int_{V_i}^{V_f} q(\Phi + \psi h') dV \tag{15}$$

In turn, the distance flown horizontally X is approximated as

$$X = \int_{V_i}^{V_f} \frac{h'}{\sin \theta} \, dV = \int_{V_i}^{V_f} V(\Phi + \psi h') dV \tag{16}$$

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This, evidently, only holds for trajectories which are (in the average) not too steep.

A linear combination of the integrals (14), (15), and (16) is now considered

$$I = K_1 t + K_2 Q + K_3 X \tag{17}$$

where K_1, K_2, and K_3 are appropriate constants, and the functional form indicated below is obtained

$$I = \int_{V_i}^{V_f} f(\Phi + \psi h') dV \tag{18}$$

where

$$f = K_1 + K_2 q + K_3 V \tag{19}$$

The following Lagrange problem is formulated: Among the transfinite set of functions $h(V)$ and $\theta(V)$ which are solutions of the differential equation (5), to determine the special set, such that the functional form (18) is extremized. The end conditions prescribe (for instance) the initial and final values for velocity and altitude. The end values for θ, on the contrary, cannot be prescribed; in view of the analytical nature of the problem, they are a consequence of the set of Euler equations.

Clearly, the above problem is a rather general one. Among its byproducts the following particular cases can, for instance, be listed:

(1) Stationary time, free fuel consumption, free horizontal distance ($K_1 = 1$, $K_2 = K_3 = 0$)

(2) Stationary fuel consumption, free time, free horizontal distance ($K_1 = K_3 = 0$, $K_2 = 1$)

(3) Stationary distance, free time, free fuel consumption ($K_1 = K_2 = 0$, $K_3 = 1$)

(4) Stationary time, free fuel consumption, given horizontal distance ($K_1 = 1$, $K_2 = 0$, $K_3 \neq 0$) W117

(5) Stationary fuel consumption, free time, given horizontal distance ($K_1 = 0$, $K_2 = 1$, $K_3 \neq 0$)

(6) Stationary time, given fuel consumption, given horizontal distance ($K_1 = 1$, $K_2 \neq 0$, $K_3 \neq 0$)

Euler Equations

A variable Lagrange multiplier $\mu(V)$ is now introduced and the following expression, denominated fundamental function, formed

$$F = f(\Phi + \psi h') + \mu\left(\frac{h'}{V \sin \theta} - \Phi - \psi h'\right) \tag{20}$$

Since there are two unknown functions, the extremal properties of the desired optimum trajectory are described in terms of two Euler equations, which can be written as follows

$$\frac{d}{dV}\left(\frac{\partial F}{\partial \theta'}\right) - \frac{\partial F}{\partial \theta} = 0 \tag{21}$$

$$\frac{d}{dV}\left(\frac{\partial F}{\partial h'}\right) - \frac{\partial F}{\partial h} = 0 \tag{22}$$

leading to

$$\frac{\mu h' \cos\theta}{V \sin^2\theta} = 0 \tag{23}$$

$$\frac{d}{dV}\left[\psi(f - \mu) + \frac{\mu}{V \sin\theta}\right] - \frac{\partial f}{\partial h}(\Phi + \psi h') + (\mu - f)\left(\frac{\partial \Phi}{\partial h} + \frac{\partial \psi}{\partial h}h'\right) = 0 \tag{24}$$

The Euler equation (23) is particularly interesting because it shows that the solution arc of the present variational problem is discontinuous, being composed of subarcs

$$\cos\theta = 0 \tag{25}$$

and subarcs

$$\mu = 0 \tag{26}$$

Equation (25) represents either a vertical zoom or a vertical dive, while equation (26) is representative of a flight condition with a continuously variable path inclination. In this connection, notice that for $\mu = 0$, equation (24) yields

$$\frac{d}{dV}(\psi f) - \frac{\partial(f\Phi)}{\partial h} - \frac{\partial(f\psi)}{\partial h}h' = 0 \tag{27}$$

In consideration of the fact that

$$\frac{d(\psi f)}{dV} = \frac{\partial(\psi f)}{\partial V} + \frac{\partial(\psi f)}{\partial h}h' \tag{28}$$

the following fundamental result is obtained

$$\frac{\partial(\psi f)}{\partial V} - \frac{\partial(f\Phi)}{\partial h} = 0 \tag{29}$$

The latter can also be, more explicitly, rewritten as

$$-EV\left(K_2\frac{\partial q}{\partial V} + K_3\right) + K_2E\frac{V^2}{g}\frac{\partial q}{\partial h} + \left(K_1 + K_2q + K_3V\right)\left(E + V\frac{\partial E}{\partial V} - \frac{V^2}{g}\frac{\partial E}{\partial h}\right) = 0 \qquad (30)$$

Particular Problems

The above equation has the merit of being general; it holds for all problems where an extremum condition (combined or not with an isoperimetric condition) is imposed on the time or the fuel consumed or the horizontal distance flown by the aircraft. Under particular circumstances, equation (30) simplifies as follows:

(1) Stationary time, free fuel consumption, free horizontal distance $\left(K_1 = 1,\ K_2 = K_3 = 0\right)$

$$E + V\frac{\partial E}{\partial V} - \frac{V^2}{g}\frac{\partial E}{\partial h} = 0 \qquad (31)$$

(2) Stationary fuel consumption, free time, free horizontal distance $\left(K_1 = 0,\ K_2 = 1,\ K_3 = 0\right)$

$$EV\left(-\frac{\partial q}{\partial V} + \frac{V}{g}\frac{\partial q}{\partial h}\right) + q\left(E + V\frac{\partial E}{\partial V} - \frac{V^2}{g}\frac{\partial E}{\partial h}\right) = 0 \qquad (32)$$

(3) Stationary distance, free time, free fuel consumption $\left(K_1 = K_2 = 0,\ K_3 = 1\right)$

$$\frac{\partial E}{\partial V} - \frac{V}{g}\frac{\partial E}{\partial h} = 0 \qquad (33)$$

(4) Stationary time, free fuel consumption, given horizontal distance $\left(K_1 = 1,\ K_2 = 0,\ K_3 \neq 0\right)$

$$-K_3EV + \left(1 + K_3V\right)\left(E + V\frac{\partial E}{\partial V} - \frac{V^2}{g}\frac{\partial E}{\partial h}\right) = 0 \qquad (34)$$

(5) Stationary fuel consumption, free time, given horizontal distance $\left(K_1 = 0,\ K_2 = 1,\ K_3 \neq 0\right)$

$$EV\left(-\frac{\partial q}{\partial V} - K_3 + \frac{V}{g}\frac{\partial q}{\partial h}\right) + \left(q + K_3V\right)\left(E + V\frac{\partial E}{\partial V} - \frac{V^2}{g}\frac{\partial E}{\partial h}\right) = 0 \qquad (35)$$

Notice that for problems (4) and (5), that is, for problems of the isoperimetric type, the term K_3 has the meaning of a constant Lagrange multiplier. The particular value of K_3 associated with a given problem must be calculated on the basis of the established isoperimetric condition and of the prescribed boundary conditions.

ANALYTICAL SOLUTIONS FOR VARIABLE PATH INCLINATION SUBARC

Under particular hypotheses concerning thrust and drag, analytical solutions can be obtained for the variable path inclination subarc.

Atmospheric Properties

The atmosphere in which the aircraft is flying is represented by means of the following relationships:

$$\frac{\rho}{\rho_*} = \pi^m \qquad (36)$$

$$\frac{\tau}{\tau_*} = \pi^{1-m} \qquad (37)$$

$$\frac{a}{a_*} = \pi^{\frac{1-m}{2}} \qquad (38)$$

where

$$\pi = \frac{p}{p_*} \qquad (39)$$

$$m = 1 + \frac{\alpha R}{g} \qquad (40)$$

$$\alpha = \frac{d\tau}{dh} \qquad (41)$$

In the above equations p is the absolute pressure, $R = p/\rho\tau$ the air constant, and α the derivative of the temperature with respect to altitude ($-0.003566\ {}^{o}R\ ft^{-1}$ for troposphere; $0\ {}^{o}R\ ft^{-1}$ for isothermal stratosphere).

Drag Function

A parabolic approximation is now assumed for the total drag coefficient

$$C_D = C_{Do}(M) + K(M)C_L{}^2 \tag{42}$$

The coefficients C_{Do} and K are assumed to depend only on the Mach number M. After accounting for equations (1) and (10) the drag function becomes

$$D = W\left(\frac{\pi}{\lambda}C_{Do}M^2 + \frac{\lambda}{\pi}\frac{K}{M^2}\right) \tag{43}$$

where

$$\lambda = \frac{2W}{\gamma p_* S} \tag{44}$$

Thrust and Specific Fuel Consumption

With the object of deriving simple solutions for the minimal problem, the thrust is assumed to be the product of a function $T_1(M)$ of the Mach number only times a function $T_2(\sigma)$ of the relative density only. The T_1 function is regarded as identical with the thrust T_* at the tropopause. The T_2 function is considered a power of the relative density σ

$$T = T_1(M)T_2\left(\frac{\rho}{\rho_*}\right) = T_*(M)\left(\frac{\rho}{\rho_*}\right)^x = T_*(M)\pi^{mx} \tag{45}$$

where x is a constant with typical values of $x = 0.75$ for tropospheric flight and $x = 1$ for stratospheric flight. The values of the m constant are $m \cong 0.81$ in the troposphere and $m = 1$ in the isothermal stratosphere.

With regard to the specific fuel consumption, the following function is assumed:

$$c = c_*(M)\left(\frac{\rho}{\rho_*}\right)^y = c_*(M)\pi^{my} \tag{46}$$

where y is a constant with typical values of $y = 0.15$ for tropospheric flight and $y = 0$ for stratospheric flight.

Transformation of Coordinates

The fundamental equation (30) is now transformed from the (h,V) coordinate system into the (π,M) system, where π is the relative pressure and $M = V/a$, the Mach number. With regard to the partial derivatives appearing in equation (30), use is made of the following operators:

$$\frac{\partial(\ldots)}{\partial V} = \frac{1}{a}\frac{\partial(\ldots)}{\partial M} \tag{47}$$

$$\frac{\partial(\ldots)}{\partial h} = -\frac{\gamma g}{a^2}\left[\pi\frac{\partial(\ldots)}{\partial \pi} + M\frac{m-1}{2}\frac{\partial(\ldots)}{\partial M}\right] \tag{48}$$

In view of these operators, equation (30) is rewritten as follows:

$$-K_3EV - K_2EM\left[\frac{\partial q}{\partial M}\left(1 + \gamma\frac{m-1}{2}M^2\right) + \gamma\pi M\frac{\partial q}{\partial \pi}\right] +$$

$$\left(K_1 + K_2q + K_3V\right)\left[E + M\frac{\partial E}{\partial M}\left(1 + \gamma\frac{m-1}{2}M^2\right) + \gamma\pi M^2\frac{\partial E}{\partial \pi}\right] = 0 \tag{49}$$

where

$$\frac{E}{W} = \frac{T_*}{W}\pi^{mx} - \frac{\pi}{\lambda}C_{Do}M^2 - \frac{\lambda}{\pi}\frac{K}{M^2} \tag{50}$$

$$\pi\frac{\partial(E/W)}{\partial \pi} = mx\frac{T_*}{W}\pi^{mx} - \frac{\pi}{\lambda}C_{Do}M^2 + \frac{\lambda}{\pi}\frac{K}{M^2} \tag{51}$$

$$M\frac{\partial(E/W)}{\partial M} = \frac{T_*}{W}\frac{d\log T_*}{d\log M}\pi^{mx} - \frac{\pi}{\lambda}C_{Do}M^2\left(2 + \frac{d\log C_{Do}}{d\log M}\right) + \frac{\lambda K}{\pi M^2}\left(2 - \frac{d\log K}{d\log M}\right) \quad (52)$$

$$q = q_*(M)\pi^{m(x+y)} \quad (53)$$

$$q_*(M) = T_*(M)c_*(M) \quad (54)$$

$$\pi\frac{\partial q}{\partial \pi} = q_* m(x+y)\pi^{m(x+y)} \quad (55)$$

$$M\frac{\partial q}{\partial M} = q_*\frac{d\log q_*}{d\log M}\pi^{m(x+y)} \quad (56)$$

Stationary Time, Free Fuel Consumption, Free Horizontal Distance

For $K_1 = 1$, $K_2 = K_3 = 0$, equation (49) reduces to

$$E + \gamma M^2\pi\frac{\partial E}{\partial \pi} + M\frac{\partial E}{\partial M}\left(1 + \gamma\frac{m-1}{2}M^2\right) = 0 \quad (57)$$

yielding (In view of the fact $\pi = \pi(h)$, eq. (58) is symbolically indicated as $\omega(M,h) = 0$.)

$$A(M)\pi^{1+mx} - B(M)\pi^2 + C(M) = 0 \quad (58)$$

where:

$$A = \frac{T_*}{W}\left[1 + \gamma mxM^2 + \left(1 + \gamma\frac{m-1}{2}M^2\right)\frac{d\log T_*}{d\log M}\right] \quad (59)$$

$$B = \frac{C_{Do}M^2}{\lambda}\left[3 + \gamma mM^2 + \left(1 + \gamma\frac{m-1}{2}M^2\right)\frac{d\log C_{Do}}{d\log M}\right] \quad (60)$$

$$C = \frac{\lambda K}{M^2}\left[1 + \gamma mM^2 - \left(1 + \gamma\frac{m-1}{2}M^2\right)\frac{d\log K}{d\log M}\right] \quad (61)$$

For the particular case of stratospheric flight $m = x = 1$, equation (58) can be solved in terms of the relative pressure, yielding

$$\pi = \sqrt{\frac{C}{B - A}} \tag{62}$$

where

$$A = \frac{T_*}{W}\left[1 + \gamma M^2 + \frac{d \log T_*}{d \log M}\right] \tag{63}$$

$$B = \frac{C_{Do} M^2}{\lambda}\left[3 + \gamma M^2 + \frac{d \log C_{Do}}{d \log M}\right] \tag{64}$$

$$C = \frac{\lambda K}{M^2}\left[1 + \gamma M^2 - \frac{d \log K}{d \log M}\right] \tag{65}$$

The altitude associated with the flight Mach number M in the stratosphere is given by

$$h = h_* - \frac{R\tau}{g} \log \pi = h_* + \frac{R\tau}{2g} \log \frac{B - A}{C} \tag{66}$$

Stationary Fuel Consumption, Free Time,

Free Horizontal Distance

For $K_1 = K_3 = 0$, $K_2 = 1$, equation (49) yields

$$-EM\left[\frac{\partial q}{\partial M}\left(1 + \gamma\frac{m - 1}{2}M^2\right) + \gamma M\pi\frac{\partial q}{\partial \pi}\right] +$$

$$q\left[E + M\frac{\partial E}{\partial M}\left(1 + \gamma\frac{m - 1}{2}M^2\right) + \gamma M^2\pi\frac{\partial E}{\partial \pi}\right] = 0 \tag{67}$$

an equation which can also be rewritten as equation (58), provided one defines

$$A = \frac{T_*}{W}\left[1 - \gamma m y M^2 - \left(1 + \gamma\frac{m - 1}{2}M^2\right)\frac{d \log c_*}{d \log M}\right] \tag{68}$$

$$B = \frac{C_{Do}M^2}{\lambda}\left[3 + \gamma m M^2(1 - x - y) + \left(1 + \gamma\frac{m - 1}{2}M^2\right)\frac{d \log\left(\frac{C_{Do}}{c_* T_*}\right)}{d \log M}\right] \tag{69}$$

$$C = \frac{\lambda K}{M^2}\left[1 + \gamma m M^2(1 + x + y) + \left(1 + \gamma\frac{m - 1}{2}M^2\right)\frac{d \log\left(\frac{c_* T_*}{K}\right)}{d \log M}\right] \tag{70}$$

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Concerning stratospheric flight $x = m = 1$, $y = 0$ the relative pressure-Mach number relationship is represented by equation (62) while the altitude-Mach number relationship is supplied by equation (66), where

$$A = \frac{T_*}{W}\left[1 - \frac{d \log c_*}{d \log M}\right] \tag{71}$$

$$B = \frac{C_{Do}M^2}{\lambda}\left[3 + \frac{d \log\left(\frac{C_{Do}}{c_* T_*}\right)}{d \log M}\right] \tag{72}$$

$$C = \frac{\lambda K}{M^2}\left[1 + 2\gamma M^2 + \frac{d \log\left(\frac{c_* T_*}{K}\right)}{d \log M}\right] \tag{73}$$

Stationary Horizontal Distance, Free Time,

Free Fuel Consumption

For $K_1 = K_2 = 0$, $K_3 = 1$, equation (49) yields

$$\frac{\partial E}{\partial M}\left(1 + \gamma\frac{m - 1}{2}M^2\right) + \gamma M \pi\frac{\partial E}{\partial \pi} = 0 \tag{74}$$

By simple transformations, the above equation can be again rewritten as equation (58), provided

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$$A = \frac{T_*}{W}\left[\gamma m x M^2 + \left(1 + \gamma\frac{m - 1}{2}M^2\right)\frac{d \log T_*}{d \log M}\right] \tag{75}$$

$$B = \frac{C_{Do}M^2}{\lambda}\left[2 + \gamma m M^2 + \left(1 + \gamma\frac{m - 1}{2}M^2\right)\frac{d \log C_{Do}}{d \log M}\right] \tag{76}$$

$$C = \frac{\lambda K}{M^2}\left[2 + \gamma m M^2 - \left(1 + \gamma\frac{m - 1}{2}M^2\right)\frac{d \log K}{d \log M}\right] \tag{77}$$

With regard to stratospheric flight, equations (62) and (66) hold, where

$$A = \frac{T_*}{W}\left[\gamma M^2 + \frac{d \log T_*}{d \log M}\right] \tag{78}$$

$$B = \frac{C_{Do}M^2}{\lambda}\left[2 + \gamma M^2 + \frac{d \log C_{Do}}{d \log M}\right] \tag{79}$$

$$C = \frac{\lambda K}{M^2}\left[2 + \gamma M^2 - \frac{d \log K}{d \log M}\right] \tag{80}$$

Comments on Solutions

(A) Consider for the sake of discussion, the minimum time problem (free fuel consumption, free horizontal distance) and transform equation (31) from the (h,V) plane into the (h_e,V) plane where h_e is a new variable defined as

$$h_e = h + \frac{V^2}{2g} \tag{81}$$

and termed energy height. The associated result is that

$$\left[\frac{\partial(EV)}{\partial V}\right]_{h_e=\text{Constant}} = 0 \tag{82}$$

The above equation constitutes the analytical basis for the energy-height method (see ref. 2), in which the speed for best climb is determined as the velocity maximizing the power excess EV (available power minus required power) for constant value of the energy height h_e.

With the graphical-analytical procedure associated with the energy method several nonoptimum points (sometimes 10 to 15) must be computed, for each value of h_e, as a preliminary step toward finding the optimum operating condition.

With the analytical solutions of the present report the optimum operating technique is supplied by a straightforward computational procedure. The advantages of the above procedure should be especially evident in the case where systematic design analyses are in order, such as those required to study the effect of variations of wing loading or thrust loading on the optimum flight program.

(B) With equation (45) the thrust function is written as the product of a function $T_1(M)$ of the Mach number only times a function $T_2(\sigma)$ of the relative density only. Consider, for the sake of discussion, a constant-geometry turbojet engine operating subsonically at constant RPM of the turbine-compressor group. For the above system, the relationship (45) is rigorously true in the isothermal stratosphere. The same relationship, however, is only an approximation in the troposphere. Thus, if a more exact solution is desired for the optimum tropospheric flight program, some refinement in the analytical representation of the thrust is necessary.

A convenient method of achieving this end consists of dividing the altitude-Mach number plane into a suitable number of regions, and utilizing a relationship of the form of equation (45) in each of these regions. In general, the functions $T_1(M)$ and $T_2(\sigma)$ have a different form from region to region. The total number of regions required evidently depends on the degree of accuracy which is desired.

It must be stressed that this procedure unavoidably leads to further discontinuities in the Mach number and/or the altitude on the line of separation between one region and another. As a consequence, appropriate fairings must be introduced a posteriori in order to join the different branches of the optimum flight program.[1]

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[1]The authors feel that, with regard to the application of the present theory, two possible types of problems must be considered, namely: (1) design of an aircraft and (2) planning of flight operations for an aircraft which has already been designed. The more precise procedure outlined in (B) is, in general, not necessary for problems of type (1) but may appear desirable for some problems of type (2).

NUMERICAL EXAMPLES

In this section numerical analyses are presented for typical turbojet-powered aircraft. The effects of variations in wing loading on the solutions are studied. The examples described are confined to the problem of stationary time, with the fuel consumption and horizontal distance traveled remaining free; however, each of the remaining problems listed in the section entitled "Formulation of Lagrange Problem" may be treated in an analogous manner.

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Aerodynamic Characteristics

Two hypothetical aircraft are considered as a basis for the following analyses. They are designated as configuration A and configuration B, respectively. For both of them the drag polar is supposed to obey equation (42) and the thrust equation (45), with $x = 0.75$ in the troposphere and $x = 1$ in the stratosphere.

In relation to configuration A, the ratio C_{Do}/C_{Doo} is plotted as a function of the Mach number in figure 1(a), where C_{Doo} denotes zero-lift drag coefficient evaluated at $M = 0$. The analogous curve for configuration B is presented in figure 1(b). The primary difference between configurations A and B lies in the fact that the drag rise in the transonic region is much steeper for configuration B.

For both configurations, the ratio K/K_o as a function of the Mach number is given in figure 2, where K_o denotes induced drag factor evaluated at $M = 0$. The ratio T_*/T_{*_o} as a function of the Mach number is given in figure 3.

Computations for Configuration A

The hypothetical aircraft designated as configuration A is considered in this section. It is assumed that the low-speed aerodynamic characteristics are such that $C_{Doo} = 0.029$ and $K_o = 0.2$. The static thrust at the tropopause is supposed to be such that $T_{*_o}/W = 0.411$, implying that the ratio of sea-level static thrust to weight is $T_{oo}/W = 1$.

Variable path inclination subarc.- The subarc of the extremal solution resulting from the solution of equation (58) is now considered.

First of all, the functions $A(M)$, $B(M)$, and $C(M)$, defined by equations (59) to (61), are determined for both tropospheric and stratospheric flight conditions. These functions are plotted in figures 4 to 6.

Secondly, as a computational aid, the relationship (eq. (58)) between A/B, C/B, and π in the troposphere is calculated and plotted in figure 7. It leads to a family of straight lines, one for each value of the pressure ratio π.

Notice that, for a given Mach number M, the sequence of graphical operations involved in solving equation (58) with the aid of figure 7 permits one to obtain only an approximate value for the pressure ratio π_a. Nevertheless, the inherent error can be corrected by writing the exact solution of equation (58) in the form

$$\pi = \pi_a(1 + \delta) \tag{83}$$

The correction term $\delta \ll 1$ can be computed by introducing equation (83) into equation (58) and linearizing the latter into

$$A\pi_a^{1+mx}\left[1 + \delta(1 + mx)\right] - B\pi_a^{2}(1 + 2\delta) + C \cong 0 \tag{84}$$

As a consequence, the correction δ is

$$\delta \cong \frac{A\pi_a^{1+mx} - B\pi_a^{2} + C}{2B\pi_a^{2} - A(1 + mx)\pi_a^{1+mx}} \tag{85}$$

For the particular case of wing loadings W/S of 60, 70, and 80 lb ft^{-2}, the solutions of equation (58) are plotted in figure 8 in the altitude-Mach number plane. As the graphs indicate, an increase in wing loading shifts the optimum speed distribution $\omega = 0$ toward the region of higher velocities. Furthermore, all the curves $\omega = 0$ inclu a branch $h =$ Constant at the tropopause, the amplitude of which increases as the wing loading increases. In the figure, the solid line AB denotes the portion of the $\omega = 0$ curve which is of interest for flight operations, insofar as it lies in the region of the altitude-Mach number plane where $T - D > 0$. The lines DBE consisting of long dashes represent the geometrical locus of the points along which the thrust T equals the drag D. Finally, the lines CB consisting of short dashes indicate the portion of the $\omega = 0$ curve which is of no interest for flight operations, insofar as it lies in the region of the altitude-Mach number plane where $T - D < 0$.

Parenthetically, it is to be stated that while the solid line AB must be flown in the sense of increasing speeds, the short dotted line CB can be flown only in the opposite sense. Thus, while AB is a solution of minimum time, the curve CB is a solution of maximum time.

Typical extremal trajectory.- A typical[2] complete extremal trajectory is shown in the altitude-Mach number plane in figure 9, where I denotes the initial flight condition, and F the final flight condition. The subarcs IG and LF are to be flown in a vertical dive, in accordance with equation (25). The subarc GHKL is flown along the $\omega = 0$ curve.

Notice that the particular extremal path sketched in figure 9 embodies several corner points, that is, several points where the function $h(M)$ shows an angular discontinuity. The latter, in turn, is the origin of a jump in the function $\theta(M)$. Clearly, the above discontinuities are not physically achievable in flight; they appear only as the consequence of the particular hypotheses (namely, the neglect of centripetal accelerations) under which the present problem has been treated. Therefore, in practical applications, the joining together of the separate subarcs is to be accomplished by means of appropriate fairings, qualitatively indicated by the dashed lines in the figure. The fairing in question must be consistent with the structural capacity of the aircraft, with the physiological ability of the pilot to withstand acceleration and, furthermore, with the thrust-drag characteristics of the aircraft.

Computations for Configuration B

The hypothetical aircraft designated as configuration B (fig. 1(b)) is now considered. This configuration is characterized by the following parameters: $C_{Doo} = 0.023$, $K_o = 0.2$, and $T_{*o}/W = 0.411$ $\left(T_{oo}/W = 1\right)$.

Variable path inclination subarc.- The functions $A(M)$ and $C(M)$ for configuration B are identical with those for configuration A and are given by figures 4 and 6. The function $B(M)$ for configuration B is presented in figure 10 for the troposphere and stratosphere.

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[2]All possible combinations of subarcs which may arise from the solution of the boundary-value problem have been indicated in figure 1 of reference 1.

The solutions of equation (58) in the altitude-Mach number plane for wing loadings of 60, 70, and 80 lb ft^{-2} are plotted in figure 11.

A comparison between figures 8 and 11 immediately yields the principal difference between configurations A and B: configuration A (mild drag gradient in the transonic region) yields a single-valued relationship in the altitude-Mach number plane, while configuration B (strong drag gradient in the transonic region) yields a multiple-valued relationship in certain regions of the altitude-Mach number plane.

This implies, therefore, that associated with configuration B there are three branches of the solution: a subsonic, a transonic, and a supersonic branch. It can be shown that the portion of the transonic branch embedded in the region of the altitude-Mach number plane where $T - D > 0$ is associated with a maximum time trajectory.

The following symbology has been used in figure 11. The solid lines AB and EG denote the portions of the $\omega = 0$ curve which are of interest for flight operations. They lie in the region of the altitude-Mach number plane where $T - D > 0$ and, furthermore, they minimize the climbing time, as can be shown by Green's theorem (ref. 3).

The line HBDGK consisting of long dashes represents the geometrical locus of the points along which $T - D = 0$

The lines BC, CD, and GL (short dashes) are embedded in the region of the altitude-Mach number plane where $T - D < 0$ and have, therefore, little interest for flight operations. They can only be flown in the sense of decreasing altitudes. Furthermore, the two lines CB and LG are associated with maximum time trajectories as Green's theorem (ref. 3) shows.

Remark.- The exact solution of possible types of boundary-value problems arising with configurations of type B may be the origin, in practice, of important analytical difficulties, of the same kind encountered by the senior writer in reference 4.

The difficulties in question are essentially concerned with the determination of the sequence of subarcs $\omega = 0$ and $\cos\theta = 0$ which form an extremal arc.

CONCLUSIONS

The climbing technique for a turbojet-powered aircraft is analyzed from the standpoint of extremizing the time required, the fuel consumed, or the horizontal distance flown by the aircraft in flying from one speed-altitude combination to another.

Trajectories minimizing any one of the above three quantities are investigated for the case where each of the other two is either free of choice or prescribed. It is shown that the totality of extremal arcs consists of a number of constant path inclination subarcs and a number of variable path inclination subarcs.

With regard to the latter, solutions in a closed form are developed, thus bypassing the graphical-analytical procedure known as energy-height method. It is shown that the altitude-Mach number relationship can be either a single-valued type or a multiple-valued type, depending upon the type of drag polar utilized. In the latter case, three branches of the solution generally exist: a subsonic branch; a transonic branch, being of no interest; and a supersonic branch.

Purdue University,
Lafayette, Ind., July 8, 1958.

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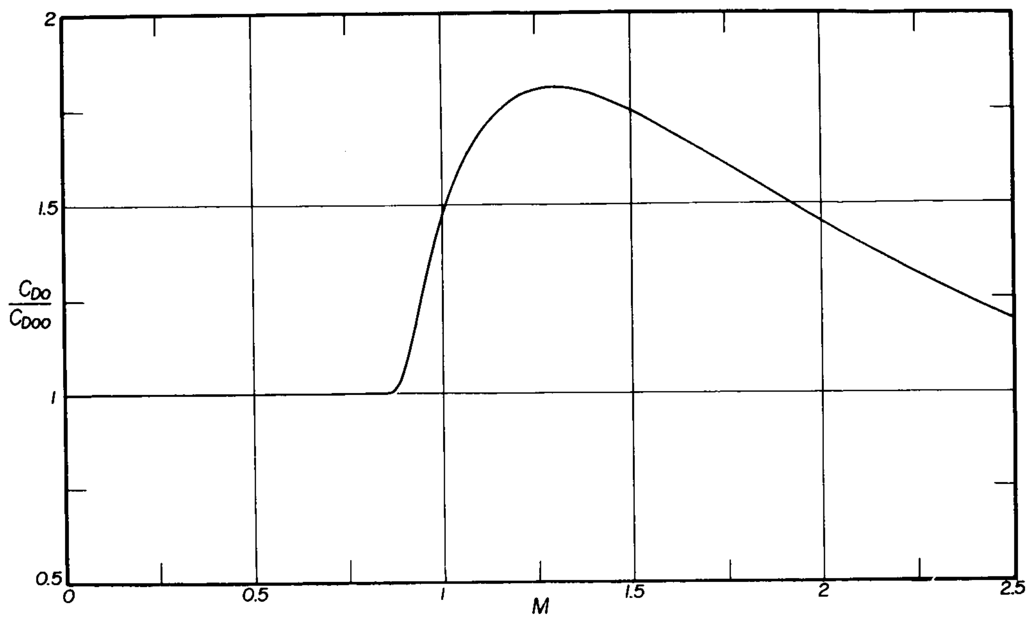
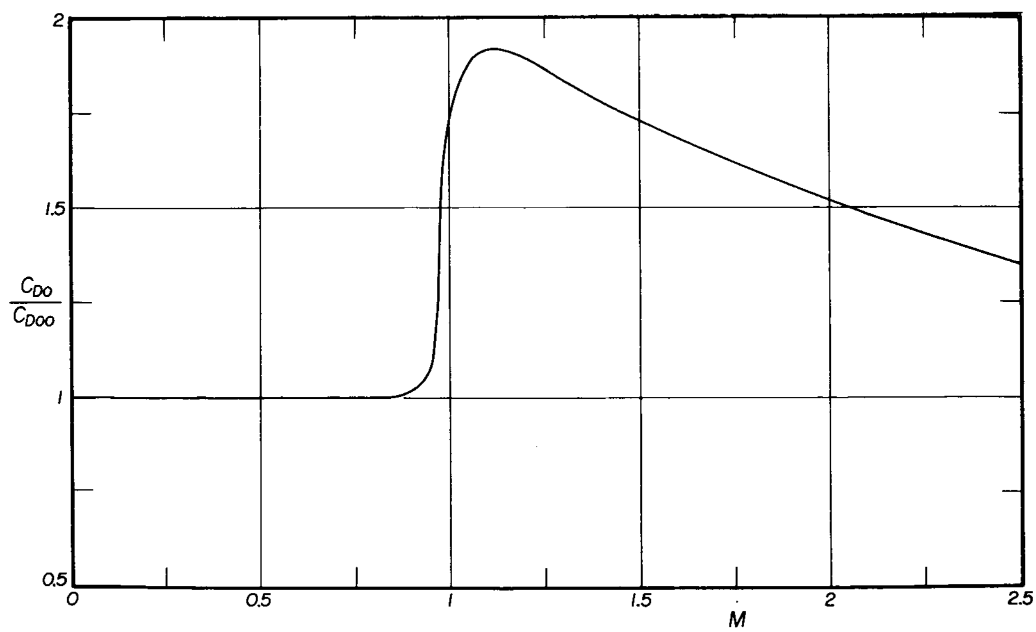
2. Rutowski, E. S.: Energy Approach to the General Aircraft Performance Problem. Jour. Aero. Sci., vol. 21, no. 3, Mar. 1954, pp. 187-195.

3. Miele, A.: General Solutions of Optimum Problems in Non-Stationary Flight. NACA TM 1388, 1955.

4. Miele, A.: An Extension of the Theory of the Optimum Burning Program for the Level Flight of a Rocket-Powered Aircraft. Tech. Note 56-302, Contract AF-18(603)-69, Office Sci. Res. and Purdue University, June 1956.



(a) Configuration A.



(b) Configuration B.

Figure 1.- Zero-lift drag coefficient as a function of Mach number.

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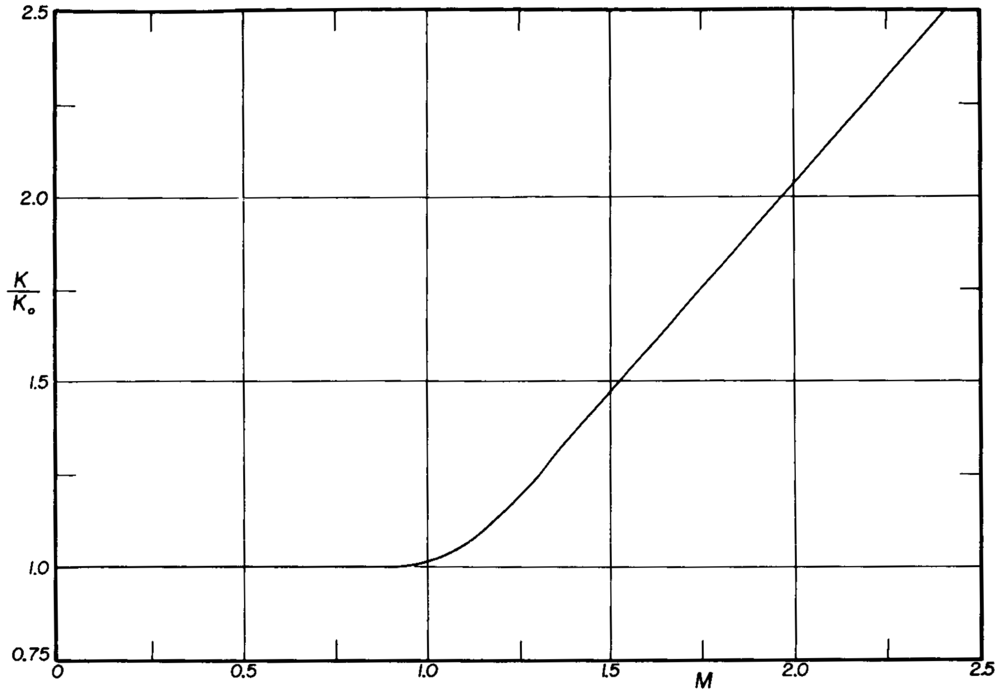


Figure 2.- Ratio of induced drag coefficient to square of lift coefficient as a function of Mach number (configurations A and B).

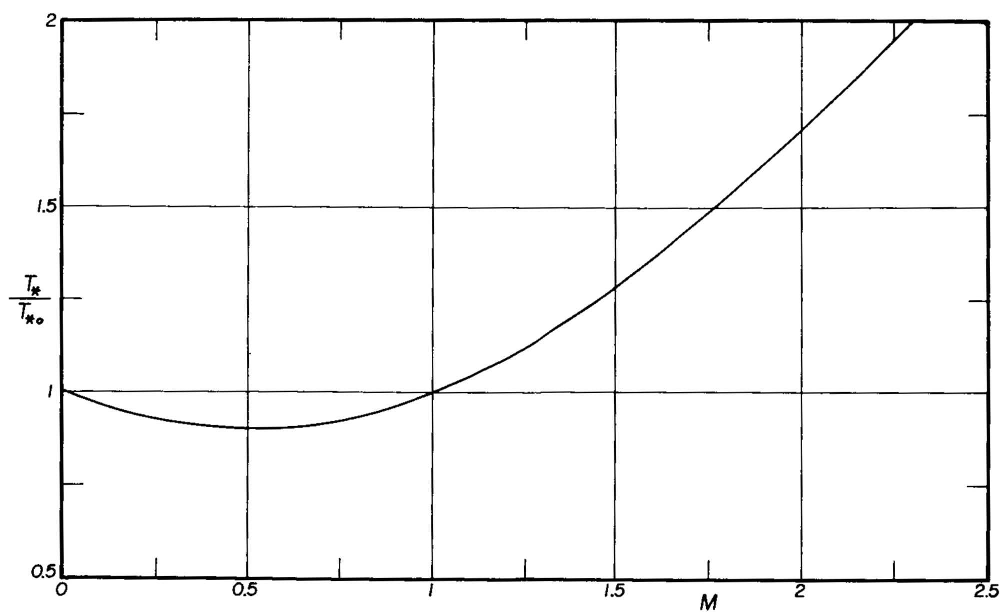
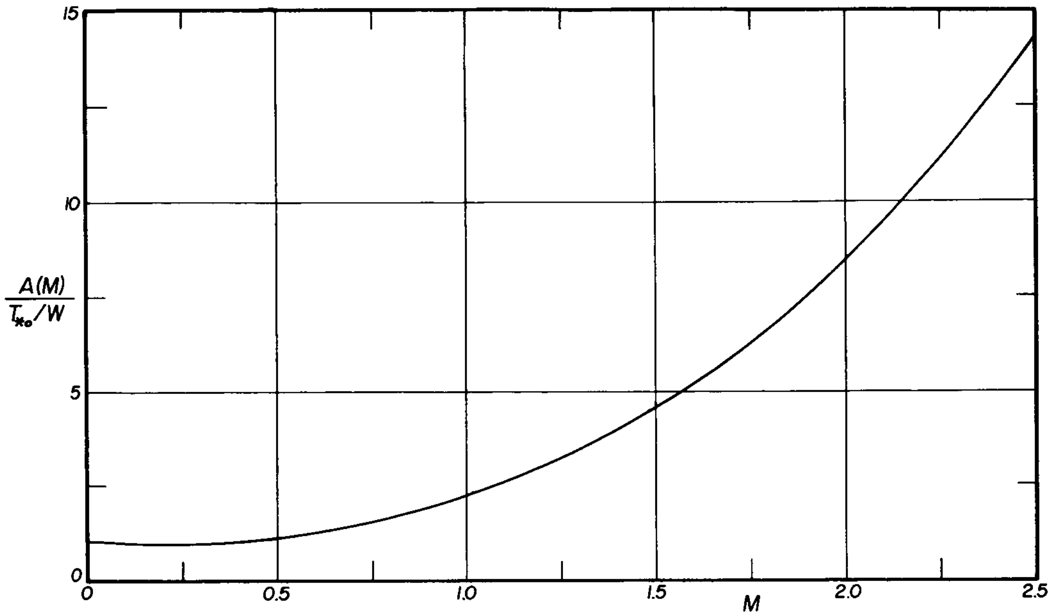
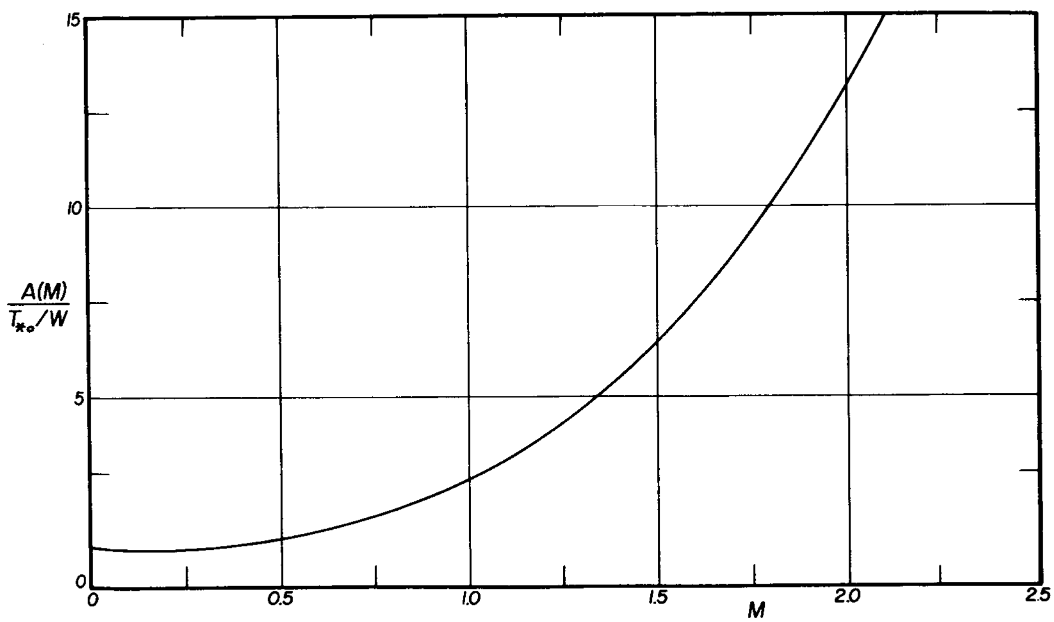


Figure 3.- Thrust at tropopause as a function of Mach number (configurations A and B).



(a) Tropospheric flight.



(b) Stratospheric flight.

Figure 4.- The function $A(M)$ for configurations A and B.

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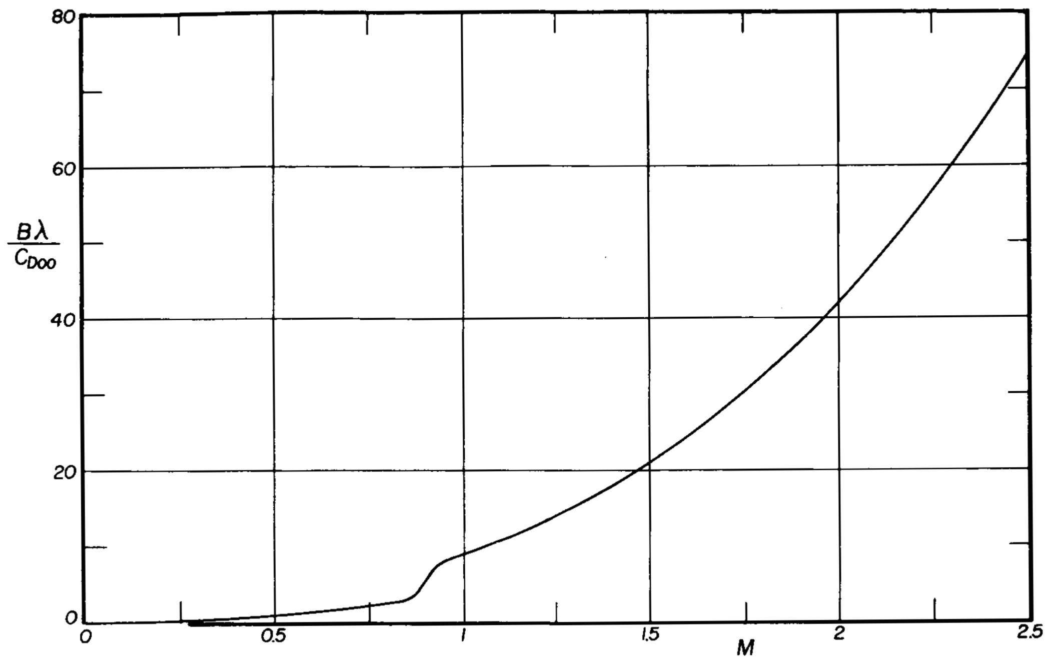
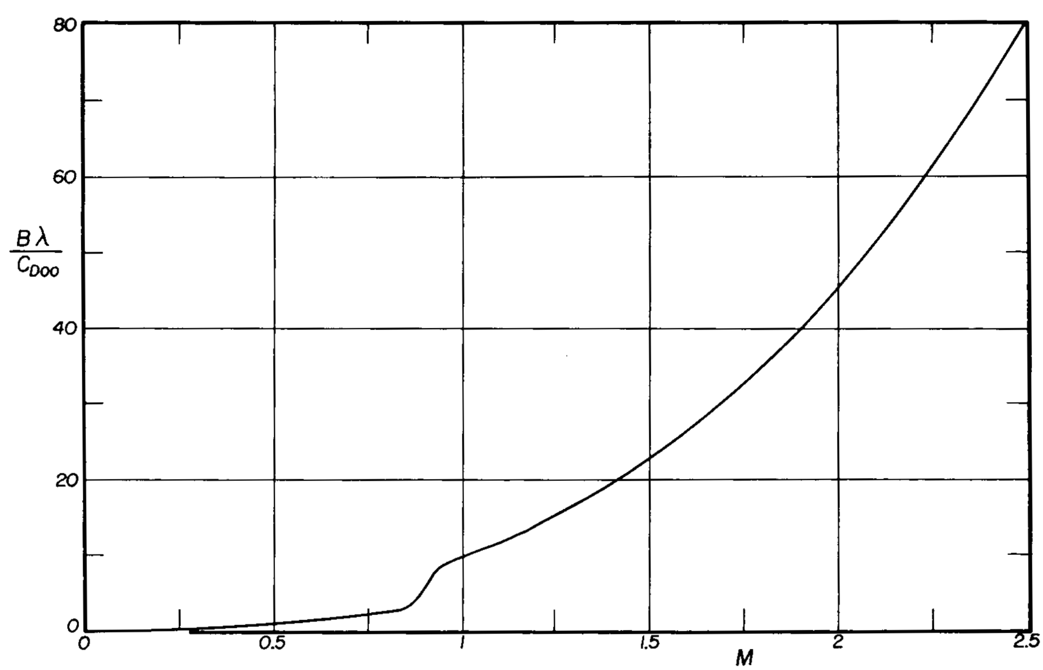
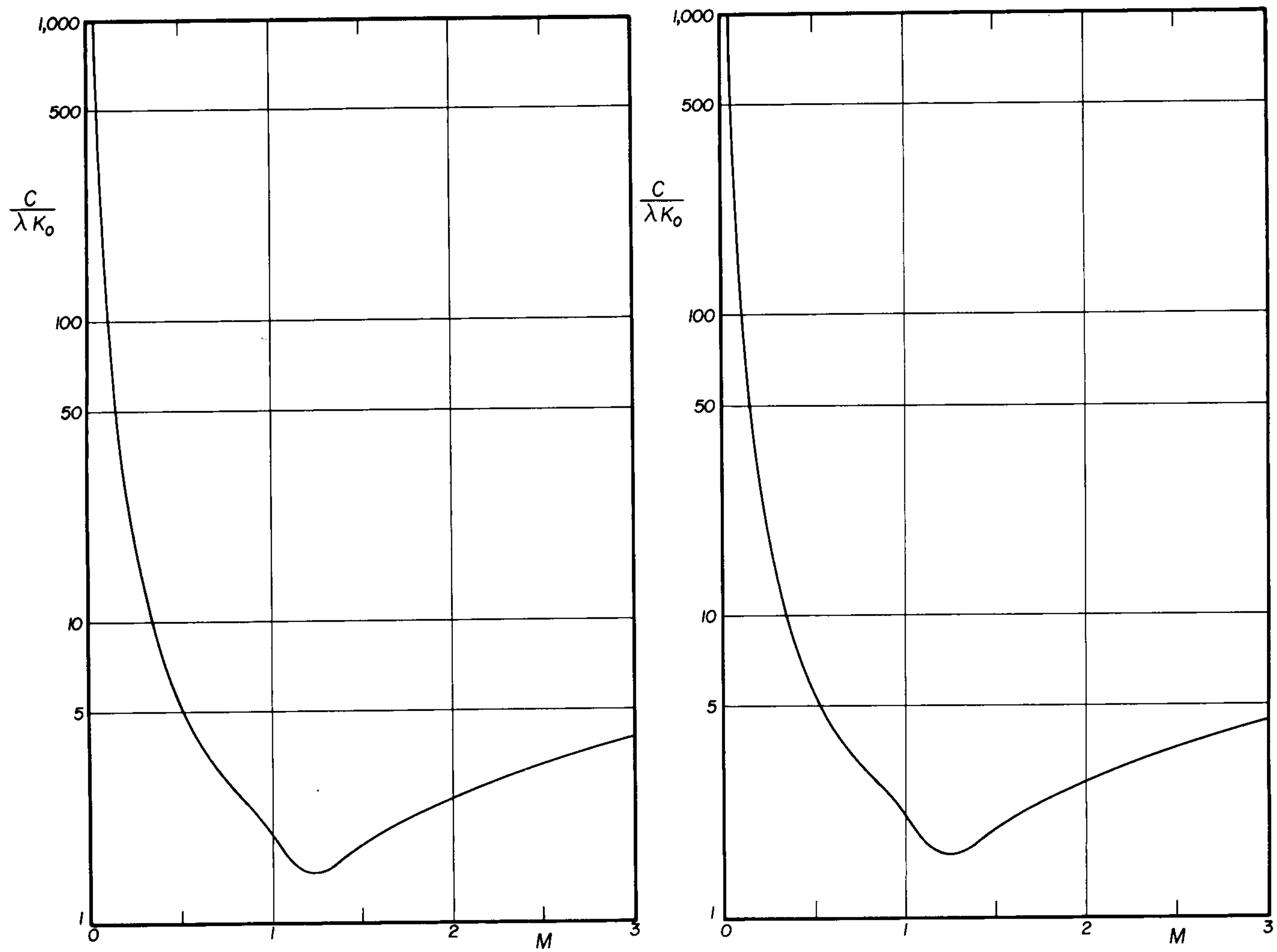


(a) Tropospheric flight.



(b) Stratospheric flight.

Figure 5.- The function B(M) for configuration A.



(a) Tropospheric flight.

(b) Stratospheric flight.

Figure 6.- The function C(M) for configurations A and B.

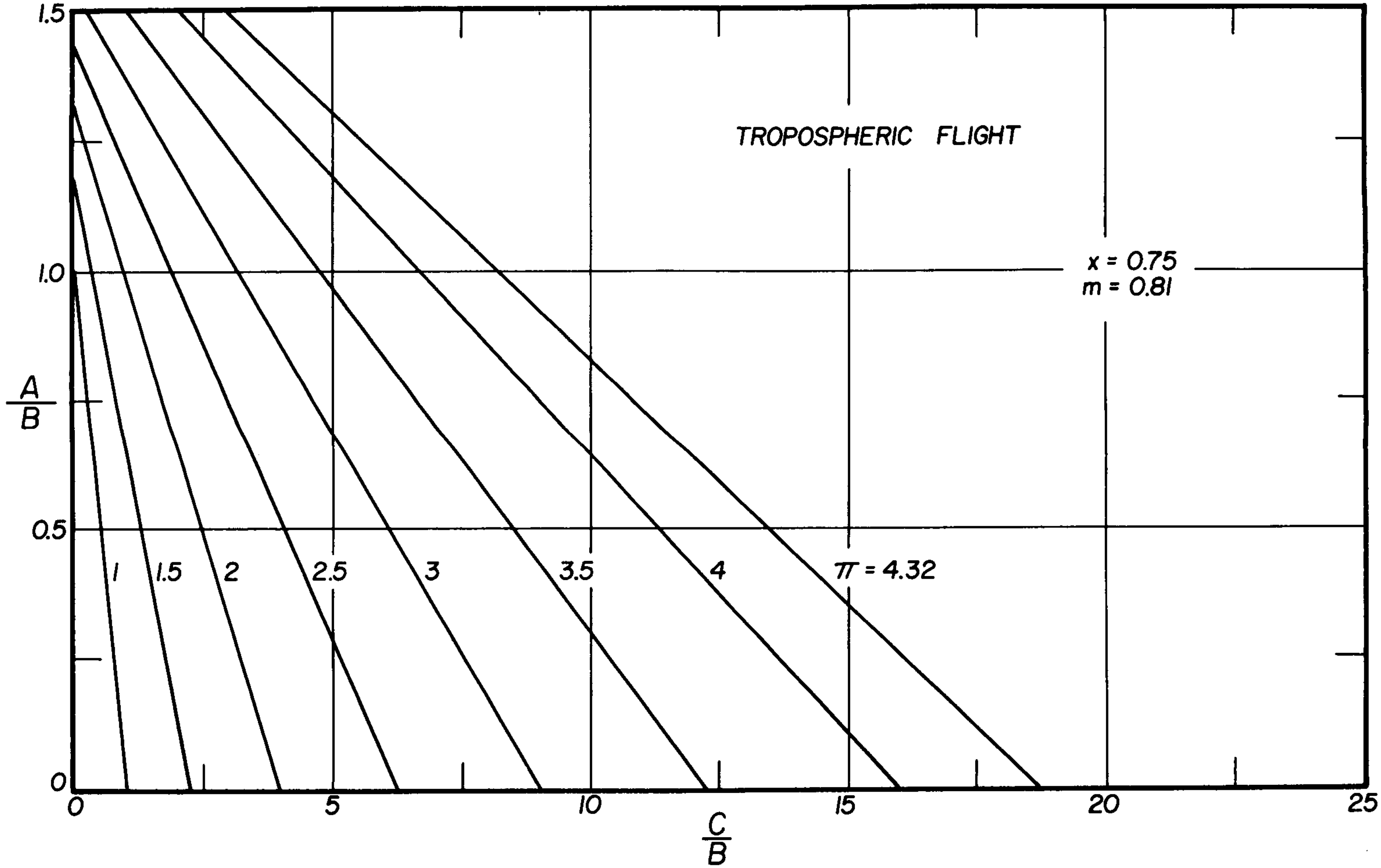
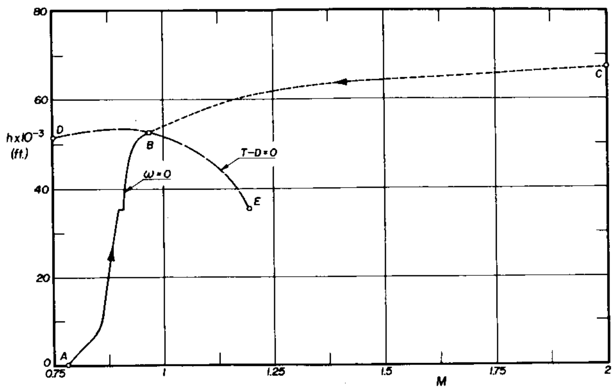
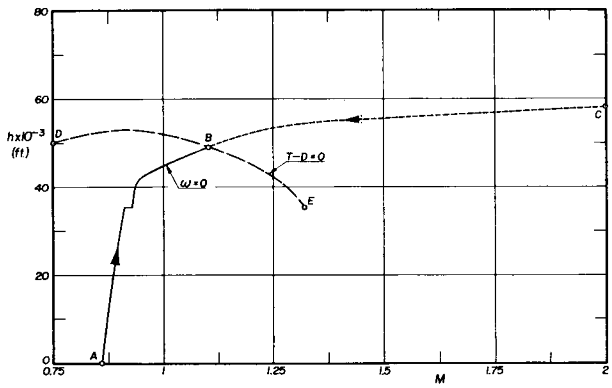


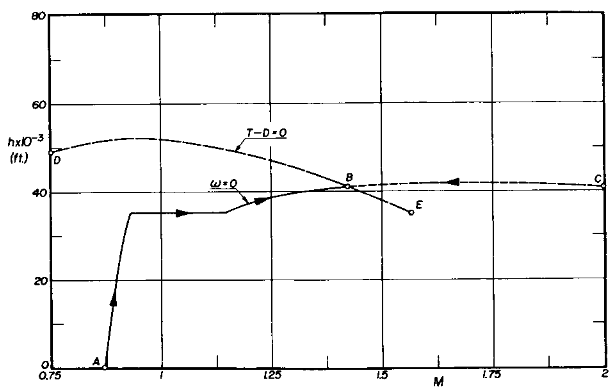
Figure 7.- Generalized graph for computation of variable path inclination subarc in troposphere.



(a) W/S = 60 $lb\text{-}ft^{-2}$.



(b) W/S = 70 $lb\text{-}ft^{-2}$.



(c) W/S = 80 $lb\text{-}ft^{-2}$.

Figure 8.- Altitude-Mach number relationship at points of variable path inclination subarc (configuration A; minimum time).

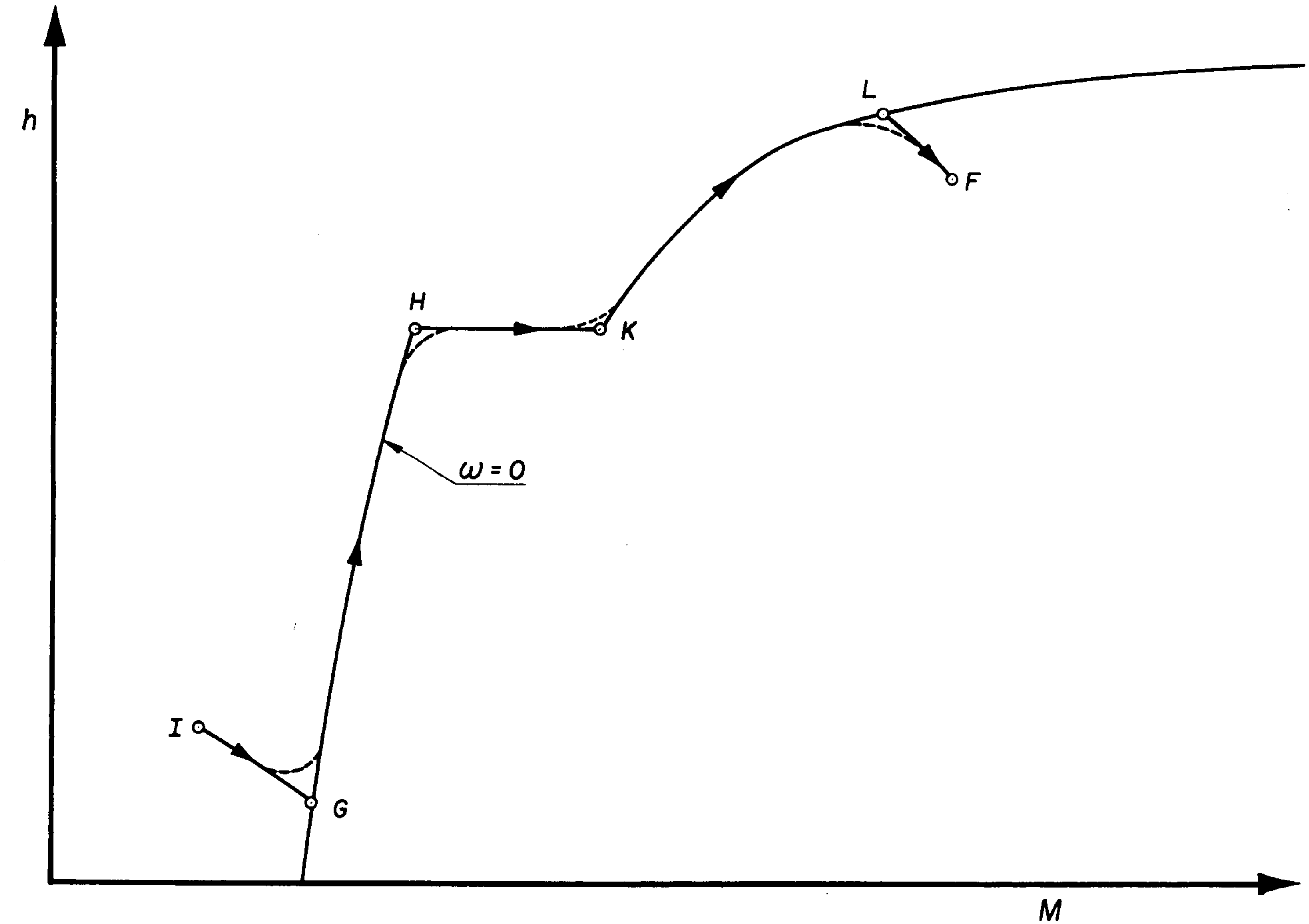
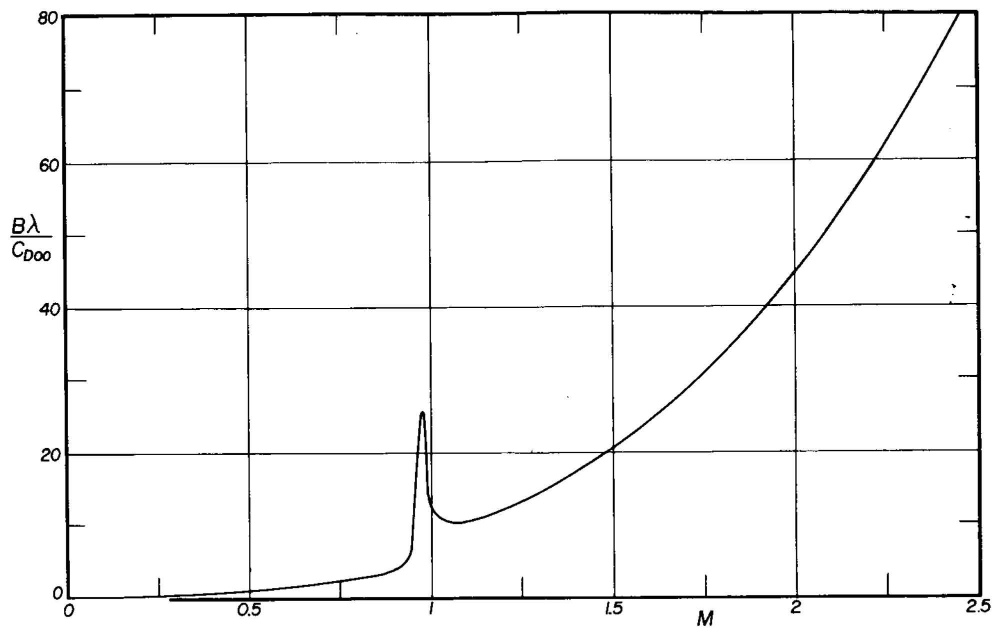
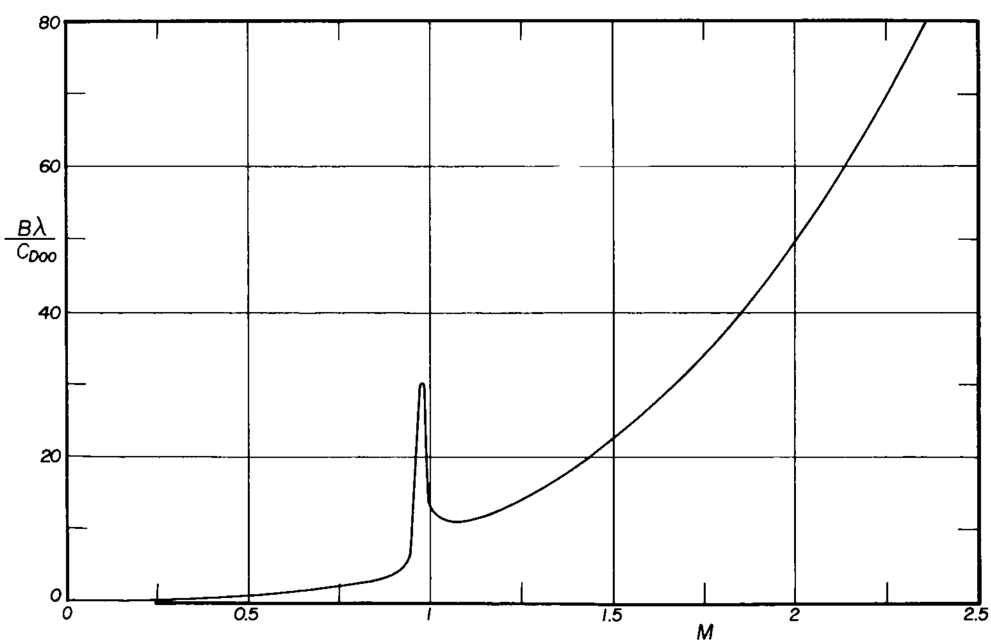


Figure 9.- Particular type of extremal arc (configuration A; minimum time).



(a) Tropospheric flight.



(b) Stratospheric flight.

Figure 10.- The function B(M) for configuration B.

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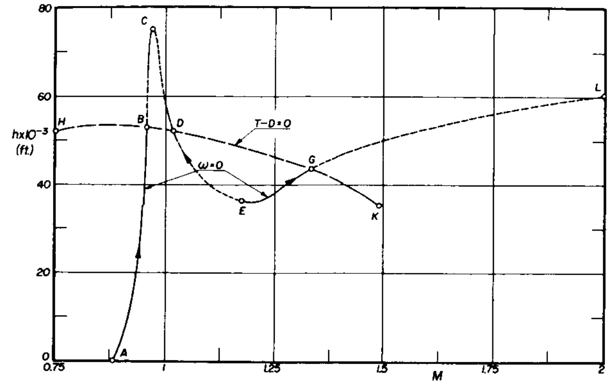
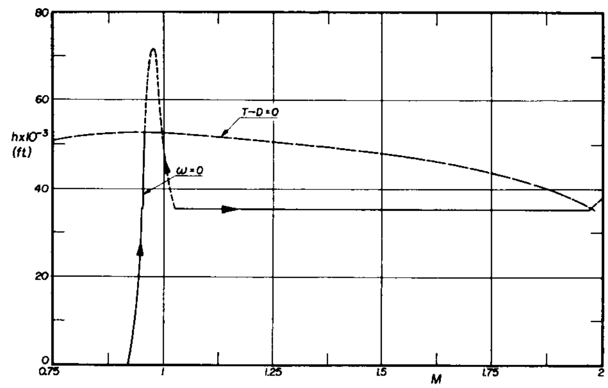
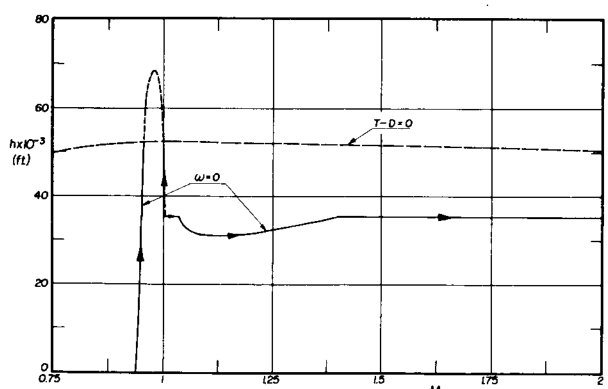


(a) W/S = 60 $lb\text{-}ft^{-2}$.



(b) W/S = 70 $lb\text{-}ft^{-2}$.



(c) W/S = 80 $lb\text{-}ft^{-2}$.

Figure 11.- Altitude-Mach number relationship at points of variable path inclination subarc (configuration B; minimum time).